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On the Order of a Restricted System of Equations.

By F. F. Decker.

I. *Introduction.*

§ 1. The conditions under which two rational integral equations of degree μ and ν respectively will have k common roots may be found as follows:* Take

$$f_1(x)\equiv a_0x^{\mu}+a_1x^{\mu-1}+\dots\dots+a_rx^{\mu-r}+\dots\dots+a_{\mu}=0$$

and

$$f_2(x)\equiv b_0x^{\nu}+b_1x^{\nu-1}+\dots\dots+b_rx^{\nu-r}+\dots\dots+b_{\nu}=0,$$

having k common roots $\alpha_1,\alpha_2,\dots\dots,\alpha_k$. They may be written

$$f_1(x)\equiv c_0(x-\alpha_1)(x-\alpha_2)\dots\dots(x-\alpha_k)\phi_1(x)$$

and

$$f_2(x)\equiv c_0(x-\alpha_1)(x-\alpha_2)\dots\dots(x-\alpha_k)\phi_2(x),$$

where

$$\phi_1(x)\equiv A_0x^{\mu-k}+A_1x^{\mu-k-1}+\dots\dots+A_rx^{\mu-k-r}+\dots\dots+A_{\mu-k}$$

and

$$\phi_2(x)\equiv B_0x^{\nu-k}+B_1x^{\nu-k-1}+\dots\dots+B_rx^{\nu-k-r}+\dots\dots+B_{\nu-k}.$$

Consequently,

$$f_1(x)\cdot\phi_2(x)\equiv f_2(x)\cdot\phi_1(x).$$

By equating the coefficients of like powers of x in the two members of this equation, there result $l-k+1$ equations linear and homogeneous in the $l-2(k-1)$ quantities $A_0, A_1,\dots\dots, A_{\mu-k}, B_0, B_1,\dots\dots, B_{\nu-k}$, $(l=\mu+\nu)$, all of which must be satisfied by the coefficients $a_0, a_1,\dots\dots, a_{\mu}, b_0, b_1,\dots\dots, b_{\nu}$, as follows:

$$\begin{array}{llr}
a_0B_0 & -b_0A_0 & =0,\\
a_1B_0+a_0B_1 & -b_1A_0-b_0A_1 & =0,\\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots & ,\\
a_hB_0+a_{h-1}B_1+\dots\dots+a_{h-\nu+k}B_{\nu-k} & -b_hA_0-b_{h-1}A_1-\dots\dots-b_{h-\mu+k}A_{\mu-k} & =0,\\
\dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots & ,\\
a_{\mu}B_0+a_{\mu-1}B_1+\dots\dots+a_{\mu-\nu+k}B_{\nu-k} & \dots\dots\dots\dots\dots\dots\dots & ,\\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots & ,\\
a_{\mu}B_{\nu-k} & \dots\dots-b_{\nu}A_{\mu-k} & =0.
\end{array}$$

The conditions resulting from eliminating the quantities $A_0, A_1,\dots\dots, A_{\mu-k}$,

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* Euler, "Berlin Memoirs," 1764, p. 96; "Histoire de l'Acad. de Paris," 1764, p. 298.

$B_0, B_1, \ldots, B_{\nu-k}$ from each set of $l-2(k-1)$ equations in turn may evidently be written in the form

$$\left\Vert \begin{array}{llllll} a_0\, a_1 \ldots\ldots a_\mu\, 0 & 0 & \ldots\ldots & & & 0 \\ 0\;\; a_0 \ldots\ldots & a_\mu\, 0 & \ldots\ldots & & & 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots & \ldots \\ 0\;\; 0\;\; \ldots\ldots & & & 0\;\; a_0 \ldots\ldots a_\mu & & \\ b_0\, b_1 \ldots\ldots & & b_\nu & 0 \ldots\ldots 0 & & \\ 0\;\; b_0 \ldots\ldots & & & b_\nu\, 0 \ldots\ldots 0 & & \\ \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots & \ldots \\ 0\;\; 0\;\; \ldots\ldots b_0 & & & \ldots\ldots b_\nu & & \end{array} \right\Vert = 0,$$

or more briefly written (using a notation to be presently explained)

$$\Vert u_{mn} \Vert^{(m)} = 0,$$

where $m = l - 2(k-1)$ and $n = l - k + 1$.

§ 2. The number of conditions is ${}_nC_m$, but it seems that not all are independent; in fact, the earlier part of the present work (III–IV) will concern itself with the number of linearly independent determinants in the set.

Subsequently each element u_{rs} of the matrix $\Vert u_{mn} \Vert^{(m)}$ will be viewed as a function of degree $a_s + \alpha_r$ of a set of variables.* The degrees of the elements are thus considered to be equidifferent with respect both to rows and to columns. The order of such a matrix will be determined (V). Finally, the geometric significance of the result will be pointed out (VI).

§ 3. A treatment of the number of solutions of the restricted system of equations obtained by equating to zero all the determinants of the m-th order of the matrix $\Vert u_{mn} \Vert$ may be found in Salmon's "Modern Higher Algebra"† (fourth edition, pp. 283–313).

After treating the special cases $(m=1, n=1)$, $(m=2, n=2)$, $(m=1, n=2)$, $(m=2, n=3)$, and $(m=1, n=3)$, Salmon announces without proof for the general case the number of solutions of the system of equations herein considered as

$$\begin{aligned} &\Sigma a_1 a_2 \ldots\ldots a_{n-m+1} + \Sigma a_1 a_2 \ldots\ldots a_{n-m} \cdot \Sigma \alpha_1 \\ &+ \Sigma a_1 a_2 \ldots\ldots a_{n-m-1} (\Sigma \alpha_1^2 + \Sigma \alpha_1 \alpha_2) + \Sigma a_1 a_2 \ldots\ldots a_{n-m-2} (\Sigma \alpha_1^3 + \Sigma \alpha_1^2 \alpha_2 + \Sigma \alpha_1 \alpha_2 \alpha_3) \\ &+ \ldots\ldots, \end{aligned}$$

or

$$\sum_{k=0}^{k=n-m+1} C_{n-m+1-k} \cdot H_k,$$

where C_k is the sum of all the elementary products of weight k involving the n a's, and H_k is the complete symmetric function of weight k involving the m α's.

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* Compare Segre, "Gli ordini delle varietà che annullano dei diversi gradi estratti da una data matrice," *Rendic. R. Accad. Dei Lincei,* series 5, Vol. IX, session of October 21, 1900.

† This treatment may also be found in Salmon's "Analytic Geometry of Space" (German translation), third edition, Vol. II, pp. 585–607.

Stuyvaert* arrives at the same problem from geometric considerations, but likewise contents himself with the results for certain cases.

§ 4. The object of this investigation is to offer a theory for the general treatment of the restricted system of equations $\|u_{mn}\|^{(m)}=0$.

The method of treatment is in the first place based on the theory of the linear independence of $n-m+1$ of the determinants of $\|u_{mn}\|^{(m)}$. The method of showing the number of such linearly independent determinants indicates also a method for their selection.

Next it is shown how to express the remaining determinants linearly in terms of the linearly independent ones (IV). In that we begin with the apparent order of the system of linearly independent determinants, it is possible, by means of these linear expressions, to discover another system by whose order the apparent order of the given system must be reduced to obtain its actual order. This second system is obtained by combining with a set of equations, less in number than the number of variables and arising from the scheme of linear expressions, a number of equations equal to the deficiency and such that the new system determines values of the variables satisfying the linearly independent determinants without satisfying the complete system. The existence of such additional equations is shown generally, and a method for their derivation is developed (V).

The calculation of the order of the second system is reduced to the calculation of the order of a third system of lower order, by whose order the apparent order of the second system must be reduced. This process is continued until there is reached a simple system whose actual order can be obtained without further reduction (§ 26).

The method gives the result surmised by Salmon, as already stated.

II. *Notation and Notational Relations.*

$$\left\|\begin{matrix} u_{11} \dots\dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots\dots u_{mn} \end{matrix}\right\|^{(p)} \quad \text{or} \quad \|u_{mn}\|^{(p)}$$

denotes the aggregate of all determinants of the p-th order that can be formed from the matrix by suppressing $m-p$ rows and $n-p$ columns. That all these determinants vanish will be denoted by

$$\left\|\begin{matrix} u_{11} \dots\dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots\dots u_{mn} \end{matrix}\right\|^{(p)} = 0 \quad \text{or} \quad \|u_{mn}\|^{(p)}=0;$$

---

* M. Stuyvaert, "Cinq études de géométrie analytique," *Memoires de la Société royale des Sciences de Liége,* 3rd series, Vol. VII, 1907; see also M. Giambelli, "Le varietà rappresentate per mezzo di una matrice generica di forme," etc., *Rendic. R. Accad. Dei Lincei,* series 5, Vol. XIV, sessions of December 3 and 17, 1905.

and the order of such a system of equations by

$$O_{\|u_{mn}\|^{(p)}}.$$

$$\|_{\overline{h_1, h_2, \ldots, h_r}, k_1, k_2, \ldots, k_s} u_{mn}\|$$

signifies a matrix formed from the matrix

$$\|u_{mn}\|$$

by suppressing the columns $h_1, h_2, \ldots, h_r$ and inserting the columns $k_1, k_2, \ldots, k_s$.

Certain matrices formed from a given matrix by the suppression or addition of columns, or both, will here be defined.

$$\begin{aligned}
G_{kA} &\equiv \|_{\overline{1, 2, \ldots, k-1}} u_{mn}\|^{(m)}, & & \\
G_{kB} &\equiv \|_{\overline{1, 2, \ldots, k-2}} u_{mm}\|^{(m-k+2)}, & & n-m+3 > k > 1, \\
G_{kC} &\equiv \|_{\overline{1, 2, \ldots, k-2}, h_1, \ldots, h_{k-2}} u_{mm}\|^{(m)}, & & k > 1, \\
G_{kD} &\equiv \|_{\overline{1, 2, \ldots, k-2}, h_1, \ldots, h_{k-3}} u_{mm}\|^{(m-1)}, & & k > 2,
\end{aligned}$$

where $h_1, h_2, \ldots, h_k$ are different integers of the series $m+1, m+2, \ldots, n$. $G_k \equiv G_{kA} + G_{kB}$; that is, the system G_k is composed of all the determinants of the two systems G_{kA} and G_{kB}. Likewise $G'_k \equiv G_{kA} + G_{kC}$. It is to be noticed that $G_1 \equiv G'_1 \equiv G_{1A}$, since $G_{1B} \equiv G_{1C} \equiv 0$; that is, there is no system G_{1B} or G_{1C}.

A second set of matrices will be defined as follows:

$$\begin{aligned}
H_1 &\equiv \|u_{m, m+1}\|^{(m)}, \\
H_{2A} &\equiv \|_{\overline{2}} u_{m, m+1}\|^{(m)}, \\
H_{2B} &\equiv \|_{\overline{1}} u_{m, m+1}\|^{(m)}, \\
H_{3C} &\equiv \|_{\overline{1, 2}} u_{m, m+1}\|^{(m-1)},
\end{aligned}$$

$$H_{3A} \equiv \left\| \begin{matrix} u_{13} \ldots u_{1\,m+1} \\ u_{33} \ldots u_{3\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots u_{m\,m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{3B} \equiv \left\| \begin{matrix} u_{23} \ldots u_{2\,m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots u_{m\,m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{4C} \equiv \left\| \begin{matrix} u_{33} \ldots u_{3\,m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots u_{m\,m+1} \end{matrix} \right\|^{(m-2)},$$

$$H_{4A} \equiv \left\| \begin{matrix} u_{33}\, u_{35} \ldots u_{3\,m+1} \\ u_{43}\, u_{45} \ldots u_{4\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,3}\, u_{m\,4} \ldots u_{m\,m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{4B} \equiv \left\| \begin{matrix} u_{34} \ldots u_{3\,m+1} \\ u_{44} \ldots u_{4\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,4} \ldots u_{m\,m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{5C} \equiv \left\| \begin{matrix} u_{35} \ldots u_{3\,m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,5} \ldots u_{m\,m+1} \end{matrix} \right\|^{(m-3)},$$

. . . .

$H_{m+1A} \equiv u_{m\,m}$ or $u_{m-1, m+1}$, according as m is odd or even, $H_{m+1B} \equiv u_{m, m+1}$, $H_{m+1C} \equiv \|u_{m\,m-1}\, u_{m\,m}\|^{(1)}$ or $u_{m-1, m+1}$, according as m is odd or even. $H_k \equiv H_{kA} + H_{kB}$.

O_R represents the order of the system R.

§ 6. Certain symmetric functions also will be defined.

$$
\begin{array}{lll}
{}_{\bar h}d_l \equiv H_l(a_{h+1},\, a_{h+2},\, \ldots,\, a_m), & d_l \equiv {}_{\bar 0}d_l, & {}_{\bar h}d_0 \equiv 1, \\
{}_{\bar h}\delta_l \equiv H_l(\alpha_{h+1},\, \alpha_{h+2},\, \ldots,\, \alpha_n), & \delta_l \equiv {}_{\bar 0}\delta_l, & {}_{\bar h}\delta_0 \equiv 1, \\
{}_{\bar h}C_l \equiv \sum\limits_{h+1}^{n} \prod\limits_{i=h+1}^{i=h+l} a_i, & C_l \equiv {}_{\bar 0}C_l, & {}_{\bar h}C_0 \equiv 1, \\
{}_{\bar h}\gamma_l \equiv \sum\limits_{h+1}^{m} \prod\limits_{i=h+1}^{i=h+l} \alpha_i, & \gamma_l \equiv {}_{\bar 0}\gamma_l, & {}_{\bar h}\gamma_0 \equiv 1. \\
{}_{\bar h}K_l \equiv \sum\limits_{i=0}^{i=l} {}_{\bar h}C_i\, \delta_{l-i}, & K_l \equiv {}_{\bar 0}K_l, & {}_{\bar h}K_0 \equiv 1, \\
{}_{\bar h}J_l \equiv \sum\limits_{i=0}^{i=l} {}_{\bar h}d_i\, \gamma_{l-i}, & J_l \equiv {}_{\bar 0}J_l, & {}_{\bar h}J_0 \equiv 1.
\end{array}
$$

§ 7. In order to facilitate later calculation, certain relations will be shown to exist among the symmetric functions just defined. First it will be shown that

$$\sum_{j=0}^{j=k}(-1)^j\,\delta_{k-j}\,\gamma_j = 0\ ^{*} \tag{1}$$

by calculating T, the coefficient of $\alpha_{i_1}^{k_1}\cdot\alpha_{i_2}^{k_2}\cdot\,.\,.\,.\,.\,.\cdot\alpha_{i_q}^{k_q}$, where $\sum\limits_1^q k_1 = k$, $k_r \neq 0$ and $q \leqq k$. From the function $\delta_{k-j}\gamma_j$, $j \leqq q$, $\alpha_{i_1}^{k_1}\cdot\alpha_{i_2}^{k_2}\cdot\,.\,.\,.\,.\,.\cdot\alpha_{i_q}^{k_q}$ can be formed by selecting terms containing any j of the quantities $\alpha_{i_1}, \ldots, \alpha_{i_q}$ from γ_j and multiplying by the proper factor from δ_{k-j}. Hence its coefficient is ${}_qC_j$. If $j > q$, $\alpha_{i_1}^{k_1}\cdot\alpha_{i_2}^{k_2}\cdot\,.\,.\,.\,.\,.\cdot\alpha_{i_q}^{k_q}$ can not be formed at all, as every term of γ_j contains j α's.

$$\therefore\ T = \sum_{j=0}^{j=q}(-1)^j\,{}_qC_j = 0,$$

and

$$\therefore\ \sum_{j=0}^{j=k}\delta_{k-j}\gamma_j = \sum_{q=1}^{q=k}\ \sum_{j=0}^{j=q}(-1)^j\,{}_qC_j\,\alpha_1^{k_1}\ \ \alpha_2\ \ \ .\,.\,.\,.\cdot\alpha_q^{k_q} = 0. \tag{2}$$

§ 8. Next it will be shown that

$$\sum_{j=0}^{j=l}(-1)^j K_{l-j}J_j = 0 \tag{3}$$

by calculating S, the sum of the terms of degree $l-k$ in the a's and k in the α's. The sum of the terms of the proposed degree in $K_{l-j}J_j$ is

$$C_{l-k}\cdot\delta_{k-j}\gamma_j + C_{l-k-1}\cdot d_1\cdot\delta_{k-j+1}\cdot\gamma_{j-1} + \ldots + C_{l-k-h}\cdot d_h\,\delta_{k-j+h}\,\gamma_{j-h} + \ldots + C_{l-k-j}\,d_j\,\delta_k\,.$$

Performing the summation for each of these terms and noting that $\delta_{k-j} = 1$ when $j = k$, and that $\delta_{k-j} = 0$ when $j > k$,

$$
\begin{aligned}
S = {}& C_{l-k}\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j - C_{l-k-1}\,d_1\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j + \,.\,.\,.\,. \\
& + (-1)^h C_{l-k-h}\,d_h\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\,\gamma_j + \,.\,.\,.\,. + (-1)^{l-k}d_{l-k}\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j \\
= {}& \sum_{i=0}^{=l-k}(-1)^i C_{l-k-i}\cdot d_i\cdot\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j,
\end{aligned}
$$

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* See Burnside and Panton, "Theory of Equations," fourth edition, Vol. I, p. 179.

and from relation (1) the second Σ vanishes. Therefore

$$\sum_{j=0}^{j=l}(-1)^j K_{l-j}J_j=\sum_{k=0}^{k=l}\left[\sum_{i=0}^{i=l-k}(-1)^i C_{l-k-i}\,d_i\cdot\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j\right]=0. \qquad (4)$$

§ 9. It will also be shown that

$$K_l={}_{\bar{1}}K_{l-1}\cdot J_1-{}_{\bar{2}}K_{l-2}\cdot{}_{\bar{1}}J_2+{}_{\bar{3}}K_{l-3}\cdot{}_{\bar{2}}J_3-\ldots\ldots$$
$$+(-1)^{j-1}{}_{\bar{j}}K_{l-j}\cdot{}_{\overline{j-1}}J_j+\ldots\ldots+(-1)^l\,{}_{\overline{l-2}}J_{l-1}. \qquad (5)$$

From equation (3)

$$K_l=K_{l-1}J_1-K_{l-2}J_2+\ldots\ldots+(-1)^{j-1}K_{l-j}J_j+\ldots\ldots+(-1)^{l-1}J_l.$$

By separating the terms of K_l into two classes, according as they do or do not contain a_1,

$$K_l=a_1\,{}_{\bar{1}}K_{l-1}+{}_{\bar{1}}K_l. \qquad (6)$$

For the J's the result is $J_l={}_{\bar{1}}J_l+a_1J_{l-1}$, and by applying the formula to itself,

$$J_l={}_{\bar{1}}J_l+a_1\,{}_{\bar{1}}J_{l-1}+a_1^2\,{}_{\bar{1}}J_{l-2}+\ldots\ldots+a_1^h\,{}_{\bar{1}}J_{l-h}. \qquad (7)$$

From 5 and 6,

$$K_l=a_1\,{}_{\bar{1}}K_{l-1}+{}_{\bar{1}}K_{l-1}\cdot{}_{\bar{1}}J_1-{}_{\bar{1}}K_{l-2}\cdot{}_{\bar{1}}J_2+\ldots\ldots$$
$$+(-1)^{j-1}{}_{\bar{1}}K_{l-j}\cdot{}_{\bar{1}}J_j+\ldots\ldots+(-1)^{l-1}{}_{\bar{1}}J_{l-1}, \qquad (8)$$

and from 7,

$$K_l={}_{\bar{1}}K_{l-1}\cdot J_1-{}_{\bar{1}}K_{l-2}\cdot{}_{\bar{1}}J_2+{}_{\bar{1}}K_{l-3}\cdot{}_{\bar{1}}J_3-\ldots\ldots$$
$$+(-1)^{j-1}{}_{\bar{1}}K_{l-j}\cdot{}_{\bar{1}}J_j+\ldots\ldots+(-1)^{l-1}{}_{\bar{1}}J_{l-1}. \qquad (9)$$

By applying (6) and (7) to the $(j-1)$-th and j-th terms of the right member of (9),

$${}_{\bar{1}}K_{l-j}\cdot{}_{\bar{1}}J_j-{}_{\bar{1}}K_{l-j+1}\cdot{}_{\bar{1}}J_{j-1}=({}_{\bar{2}}K_{l-j}+a_2\cdot{}_{\bar{2}}K_{l-j-1})\,({}_{\bar{2}}J_j+a_2\cdot{}_{\bar{2}}J_{j-1}+\ldots\ldots+a_2^{j-2}\cdot{}_{\bar{2}}J_2)$$
$$-({}_{\bar{2}}K_{l-j+1}+a_2\cdot{}_{\bar{2}}K_{l-j})\,({}_{\bar{2}}J_{j-1}+a_2\cdot{}_{\bar{2}}J_{j-2}+\ldots\ldots+a_2^{j-2}\cdot{}_{\bar{2}}J_2),$$

and therefore the factor multiplying ${}_{\bar{2}}K_{l-j}$ in (9) is ${}_{\bar{2}}J_j$, and (9) becomes

$$K_l={}_{\bar{1}}K_{l-1}\cdot J_1-{}_{\bar{2}}K_{l-2}\cdot{}_{\bar{1}}J_2+{}_{\bar{2}}K_{l-3}\cdot{}_{\bar{2}}J_3-\ldots\ldots$$
$$+(-1)^{j-1}{}_{\bar{2}}K_{l-j}\cdot{}_{\bar{2}}J_j+\ldots\ldots+(-1)^{j-1}{}_{\bar{2}}J_j.$$

Repeating this reduction a sufficient number of times,

$$K_l={}_{\bar{1}}K_{l-1}\cdot J_1-{}_{\bar{2}}K_{l-2}\cdot{}_{\bar{1}}J_2+{}_{\bar{3}}K_{l-3}\cdot{}_{\bar{2}}J_3-\ldots\ldots$$
$$+(-1)^{j-1}{}_{\bar{j}}K_{l-j}\cdot{}_{\overline{j-1}}J_j+\ldots\ldots+(-1)^{l-1}{}_{\overline{l-1}}J_l. \qquad (10)$$

III. *Some Theorems on the Linear Independence of Certain of the Determinants* $\|u_{m\,n}\|^{(m)}$.

§10. THEOREM 1: In $\|u_{m\,m+1}\|^{(m)}$ there are two linearly independent determinants.

Write

$$\Delta\equiv\left|\begin{array}{l} v_{11}\,v_{12}\;\ldots\ldots v_{1\,m+1}\\ u_{11}\,u_{12}\;\ldots\ldots u_{1\,m+1}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u_{m1}\,u_{m2}\ldots\ldots u_{m\,m+1}\end{array}\right|,$$

and let V_{1k} be the cofactor of v_{1k} in Δ. Then

$$\begin{array}{ll}
u_{11} V_{11} + \quad u_{12} V_{12} + \dots + \quad u_{1\,m+1} V_{1\,m+1} = 0 & (e_1), \\
u_{21} V_{11} + \quad u_{22} V_{12} + \dots + \quad u_{2\,m+1} V_{1\,m+1} = 0 & (e_2), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & , \\
u_{m-1\,1} V_{11} + v_{m-1\,2} V_{12} + \dots + u_{m-1\,m+1} V_{1\,m+1} = 0 & (e_{m-1}), \\
u_{m\,1} V_{11} + \quad u_{m\,2} V_{12} + \dots + \quad u_{m\,m+1} V_{1\,m+1} = 0 & (e_m).
\end{array}$$

(e_1) makes it possible to express one V linearly in terms of the m others by means of determinants of the first order, showing that there are not more than m linearly independent V's. (e_1) and (e_2) make it possible to express two V's linearly in terms of the $m-1$ others by means of determinants of the second order, showing that there are not more than $m-1$ linearly independent V's. Finally, the system (e_1), (e_2), ..., (e_{m-1}) makes it possible to express $m-1$ V's linearly in terms of the two others by means of determinants of the $(m-1)$-th order, showing that there are not more than two linearly independent V's.

It might seem that the system (e_1), (e_2), ..., (e_m) would enable one to express m V's linearly in terms of the other one by means of determinants of the m-th order; but such is not the case, for when the solution is attempted an identity results, as will be shown by an attempt to solve for $V_{1\,k}$ in terms of $V_{1\,m+1}$.

$$V_{1k} = \frac{\begin{vmatrix} u_{11} \dots u_{1k-1} \; u_{1m+1} \; u_{1k+1} \dots u_{1m} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_{m1} \dots u_{mk-1} \; u_{mm+1} \; u_{mk+1} \dots u_{mm} \end{vmatrix}}{\begin{vmatrix} u_{11} \dots\dots\dots\dots\dots\dots u_{1m} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_{m1} \dots\dots\dots\dots\dots\dots u_{mm} \end{vmatrix}} V_{1m+1} = (-1)^{m-k+1} \frac{\begin{vmatrix} u_{11} \dots u_{1k-1} \; u_{1k+1} \dots u_{1m+1} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_{m1} \dots u_{mk-1} \; u_{mk+1} \dots u_{mm+1} \end{vmatrix}}{\begin{vmatrix} u_{11} \dots\dots\dots\dots\dots\dots u_{1m} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_{m1} \dots\dots\dots\dots\dots\dots u_{mm} \end{vmatrix}} V_{1m+1},$$

or

$$\Delta_{1\,m+1} V_{1\,k} = (-1)^{m-k+1} \Delta_{1\,k} V_{1\,m+1},$$

where $\Delta_{1\,k}$ is the determinant formed by deleting the k-th column of $\|u_{m\,m+1}\|$; that is,

$$\Delta_{1\,k} \equiv \|{}_{\bar{k}}u_{m\,m+1}\|^{(m)}.$$

This gives $\Delta_{1\,h} = (-1)^{1+h} V_{1\,h}$, and therefore

$$(-1)^{m+2} V_{1\,m+1} \cdot V_{1\,k} = (-1)^{m-k+1} (-1)^{k+1} V_{1\,k} \cdot V_{1\,m+1},$$

or

$$V_{1\,m+1} \cdot V_{1\,k} = V_{1\,m+1} \cdot V_{1\,k},$$

an identity.

Since the V's are numerically the determinants $\|u_{m\,m+1}\|^{(m)}$, the theorem follows.

The theorem shows that any two of the determinants may be used as the two linearly independent ones. Unless otherwise stated, the two chosen will be the two having the first $m-1$ columns common; that is,

$$\|_{\overline{m+1}}\, u_{m\; m+1}\|^{(m)} \text{ and } \|_{\overline{m}}\, u_{m\; m+1}\|^{(m)}.$$

§ 11. It will be shown by mathematical induction that the number of linearly independent determinants of the system $\|u_{m\,n}\|^{(m)}$ is $n-m+1$; that is, that all the determinants of the system can be linearly expressed in terms of $n-m+1$ of them.

Before passing from the case just given to the general case, a few special cases will be given to elucidate the process.

§ 12. Next it will be shown that

THEOREM 2: $\|u_{m\; m+2}\|^{(m)}$ can be linearly expressed in terms of

$$\left.\begin{array}{rl} \|_{\overline{m+1,\, m+2}}\, u_{m\; m+2}\|^{(m)} & 1, \\ \|_{\overline{m,\, m+2}}\, u_{m\; m+2}\|^{(m)} & 2, \\ \|_{\overline{m,\, m+1}}\, u_{m\; m+2}\|^{(m)} & 3, \end{array}\right\} I_{m,\, m+2}\,.$$

and

The determinants

$$\|_{\overline{m+2}}\, u_{m\; m+2}\|^{(m)} \tag{a}$$

and

$$\|_{\overline{m+1}}\, u_{m\; m+2}\|^{(m)} \tag{b}$$

can be so expressed; for, by theorem 1, (a) can be so expressed in terms of 1 and 2 and (b) in terms of 1 and 3.

Since the remaining matrices of $m+1$ columns contain both columns $m+1$ and $m+2$, they contain $m-1$ of the first m columns each. The one that contains the first $m-1$ columns $\|_{\overline{m}}\, u_{m\; m+2}\|^{(m)}$ is linearly expressible, according to theorem 1, in terms of 2 and 3.

For the case of one of the others,

$$\|_{\overline{h}}\, u_{m\; m+2}\|^{(m)}, \qquad h=1,\, 2,\, \ldots,\, m-1,$$

it is linearly expressible, according to theorem 1, in terms of $\|_{\overline{h,\, m+2}}\, u_{m\; m+2}\|^{(m)}$ and $\|_{\overline{h,\, m+1}}\, u_{m\; m+2}\|^{(m)}$, the first of which is included in (a), the second in (b); and (a) and (b) have already been shown to be linearly dependent on $I_{m,\, m+2}$. Thus all the determinants $\|u_{m\; m+2}\|^{(m)}$ are linearly dependent on the system $I_{m,\, m+2}$.

§ 13. Next it will be shown that

THEOREM 3: $\|u_{m\; m+3}\|^{(m)}$ can be linearly expressed in terms of

$$\left.\begin{array}{rl} \|_{\overline{m+1,\, m+2,\, m+3}}\, u_{m\; m+3}\|^{(m)} & 1, \\ \|_{\overline{m,\, m+2,\, m+3}}\, u_{m\; m+3}\|^{(m)} & 2, \\ \|_{\overline{m,\, m+1,\, m+3}}\, u_{m\; m+3}\|^{(m)} & 3, \\ \|_{\overline{m,\, m+1,\, m+2}}\, u_{m\; m+3}\|^{(m)} & 4, \end{array}\right\} I_{m,\, m+3}\,.$$

and

The determinants

$$\|_{\overline{m+3}}\, u_{m\;m+3}\|^{(m)} \qquad \text{(a)}$$

and

$$\|_{\overline{m+2}}\, u_{m\;m+3}\|^{(m)} \qquad \text{(b)}$$

can be so expressed, for, by theorem 2, (a) can be so expressed in terms of 1, 2 and 3, and (b) in terms of 1, 2 and 4.

Since the remaining matrices of $m+2$ columns contain both columns $m+2$ and $m+3$, they contain also m of the first $m+1$ columns each. The one that contains the first m,

$$\|_{\overline{m+1}}\, u_{m\;m+3}\|^{(m)},$$

is linearly expressible, according to theorem 2, in terms of 1, 3 and 4.

For the case of one of the others, $\|_{\bar{h}} u_{m\;m+3}\|^{(m)}$, $h=1, 2, \ldots, m$, it is linearly expressible, according to theorem 2, in terms of

$$\|_{\overline{h,\, m+2,\, m+3}}\, u_{m\;m+3}\|^{(m)},$$

$$\|_{\overline{h,\, m+1,\, m+3}}\, u_{m\;m+3}\|^{(m)},$$

and

$$\|_{\overline{h,\, m+1,\, m+2}}\, u_{m\;m+3}\|^{(m)},$$

the first two of which are included in (a), the third in (b); and (a) and (b) have already been shown to be linearly dependent on the system $I_{m,\,m+3}$. Thus all the determinants

$$\|u_{m\;m+3}\|^{(m)}$$

are linearly dependent on the system $I_{m,\,m+3}$.

§ 14. It will now be shown that

THEOREM 4: The determinants $\|u_{m\;m+r}\|^{(m)}$ are linearly expressible in terms of the $r+1$ determinants

$$\left.\begin{array}{rr}
\|_{\overline{m+1,\ldots\ldots,m+r}}\, u_{m\;m+r}\|^{(m)} & 1, \\
\|_{\overline{m,\, m+2,\ldots\ldots,\, m+r}}\, u_{m\;m+r}\|^{(m)} & 2, \\
\|_{\overline{m,\, m+1,\, m+3,\ldots\ldots,\, m+r}}\, u_{m\;m+r}\|^{(m)} & 3, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \ldots\ldots, \\
\|_{\overline{m,\ldots\ldots,\, m+h-1,\, m+h+1,\ldots\ldots,\, m+r}}\, u_{m\;m+r}\|^{(m)} & h+1, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \ldots\ldots, \\
\|_{\overline{m,\ldots\ldots,\, m+r-1}}\, u_{m\;m+r}\|^{(m)} & r+1,
\end{array}\right\} I_{m,\,m+r};$$

provided the expressibility is possible when r is replaced by $r-1$.

The determinants

$$\|_{\overline{m+r}}\, u_{m\;m+r}\|^{(m)} \qquad \text{(a)}$$

and

$$\|_{\overline{m+r-1}}\, u_{m\;m+r}\|^{(m)} \qquad \text{(b)}$$

can be so expressed, for, by the assumption, (a) can be so expressed in terms of 1, 2, ..., r, and (b) in terms of 1, 2, ..., $r-1$ and $r+1$.

Since the remaining matrices of $m+r-1$ columns contain both the columns

$m+r-1$ and $m+r$, each of them contains also $m+r-3$ of the first $m+r-2$ columns; that is, they are

$$\|_{\bar{h}}u_{m\,m+r}\|^{(m)}, \quad h=1,\,2,\,\ldots\,,\,m+r-2.$$

The matrix $\|_{\overline{m+r-2}}\,u_{m\,m+r}\|^{(m)}$ is linearly expressible, according to the assumption, in terms of $2,\,3,\,\ldots\,,\,r+1$.

For the case of one of the others,

$$\|_{\bar{h}}u_{m\,m+r}\|^{(m)}, \quad h=1,\,2,\,\ldots\,,\,m+r-3,$$

it is linearly expressible, according to the assumption, in terms of

$$\begin{array}{r} \|_{\overline{h,\ m+2,\ \ldots,\ m+r}}\,u_{m\,m+r}\|^{(m)}, \\ \|_{\overline{h,\ m+1,\ m+3,\ \ldots,\ m+r}}\,u_{m\,m+r}\|^{(m)}, \\ \|_{\overline{h,\ m+1,\ m+2,\ m+4,\ \ldots,\ m+r}}\,u_{m\,m+r}\|^{(m)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \|_{\overline{h,\ m+1,\ldots,\ m+k-1,\ m+k+1,\ \ldots,\ m+r}}\,u_{m\,m+r}\|^{(m)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

and $$\|_{\overline{h,\ m+1,\ldots,\ m+r-1}}\,u_{m\,m+r}\|^{(m)},$$

all but the last of which are included in (a), the last being included in (b). As (a) and (b) have already been shown to be linearly dependent on the system $I_{m,\,m+r}$, it follows that all the determinants $\|u_{m\,m+r}\|^{(m)}$ are linearly dependent on the system $I_{m,\,m+r}$.

§ 15. Theorems 1, 2 and 3 respectively state that the assumption of theorem 4 is true for the cases $r=2$, 3 and 4. Therefore by mathematical induction there follows the more general theorem

THEOREM 5: The number of linearly independent determinants among $\|u_{m\,n}\|^{(m)}$ is $n-m+1$.*

§ 16. It has been pointed out (§ 1) that the condition under which two rational integral equations of degrees μ and ν respectively will have k common roots may be written in the form

$$\|u_{m\,n}\|^{(m)}=0,$$

where $n=\mu+\nu-k+1$ and $m=\mu+\nu-2(k-1)$. It is to be noticed from theorem 5 that the number of linearly independent determinants involved is

$$n-m+1=[\mu+\nu-k+1]-[\mu+\nu-2(k-1)]+1=k.$$

IV. *A Problem on Expressibility.*

§ 17. The expressibility of all the determinants $\|u_{m\,n}\|^{(m)}$ in terms of $n-m+1$ of them having been shown, the problem of actually expressing them will be considered.

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* Compare Sommerville, *Proceedings of the Edinburgh Mathematical Society,* Vol. XXIV, November 10, 1905. See also Muir, "On Some hitherto Unproved Theorems in Determinants," *Proceedings of the Royal Society of Edinburgh* for 1890–1891.

PROBLEM 1: To express any determinant $|h_1, h_2, \ldots, h_m|$ of $\|u_{mn}\|^{(m)}$ in terms of the determinants $|1, 2, \ldots, m-1, k|$, $k=m, m+1, \ldots, n$.

The first case considered will be that in which $|h_1, h_2, \ldots, h_m|$ contains none of the first $m-1$ columns of $\|u_{mn}\|$,

$$|h_1, h_2, \ldots, h_m| \equiv \begin{vmatrix} u_{1h_1} & u_{1h_2} & \ldots & u_{1h_m} \\ u_{2h_1} & u_{2h_2} & \ldots & u_{2h_m} \\ \ldots & \ldots & \ldots & \ldots \\ u_{mh_1} & u_{mh_2} & \ldots & u_{mh_m} \end{vmatrix}.$$

Multiply the first row by W_1, the second by W_2, etc., and add to the first row the sum of those following, where W_k is the cofactor of w_k in

$$\begin{vmatrix} u_{11} & \ldots & u_{1\,m-1} & w_1 \\ u_{21} & \ldots & u_{2\,m-1} & w_2 \\ \ldots & \ldots & \ldots & \ldots \\ u_{m1} & \ldots & u_{m\,m-1} & w_m \end{vmatrix}.$$

$$|h_1, h_2, \ldots, h_m| \equiv \frac{1}{W_1 W_2 \ldots W_m} \begin{vmatrix} u_{1h_1}W_1 & u_{1h_2}W_1 & \ldots & u_{1h_m}W_1 \\ u_{2h_1}W_2 & u_{2h_2}W_2 & \ldots & u_{2h_m}W_2 \\ \ldots & \ldots & \ldots & \ldots \\ u_{mh_1}W_m & u_{mh_2}W_m & \ldots & u_{mh_m}W_m \end{vmatrix},$$

or

$$|h_1, h_2, \ldots, h_m| \equiv \frac{1}{W} \begin{vmatrix} |1, 2, \ldots, m-1, h_1| & \ldots & |1, 2, \ldots, m-1, h_m| \\ u_{2h_1} & \ldots & u_{2h_m} \\ \ldots & \ldots & \ldots \\ u_{mh_1} & \ldots & u_{mh_m} \end{vmatrix}.$$

By expanding the determinant in the right member in terms of the elements of its first row,

$$\begin{aligned} |h_1, h_2, \ldots, h_m| &\equiv \frac{1}{W_1}\,[\,|h_2, \ldots, h_m| \cdot |1, 2, \ldots, m-1, h_1| \\ &\qquad - |h_1, h_3, \ldots, h_m| \cdot |1, 2, \ldots, m-1, h_2| + \ldots] \\ &\equiv \frac{1}{W_1} \sum_{j=1}^{j=m} (-1)^{j-1} |h_{i_1}, \ldots, h_{i_{j-1}}, h_{i_{j+1}}, \ldots, h_{i_m}| \\ &\qquad \cdot |1, 2, \ldots, m-1, h_{i_j}|, \end{aligned} \tag{11}$$

where $|h_1, h_2, \ldots, h_{m-1}|$ is a determinant formed from $|h_1, h_2, \ldots, h_{m-1}, h_l|$ by suppressing the first row and the last column.

§ 18. To express $|h_1, h_2, \ldots, h_m|$, in the case under consideration, evidently $n-m+1$ linearly independent determinants, that is, a complete system, are required. To express $|h_1, h_2, \ldots, h_m|$ in terms of linearly independent determinants, in the case where it contains k of the first $m-1$ columns, the argument as given for the other case is valid, but evidently only $n-m+1-k$ of the linearly independent determinants are required, as k of the determinants in the right member of formula (11) have each two common columns and vanish identically.

§ 19. By a slight change in the details of the proof a formula similar to (11) may be obtained in which W_1 is replaced by W_k, $k=1, 2, \ldots, m$. However there follows the

COROLLARY: The expressibility is not possible if

$$\|u_{m\,m-1}\|^{(m-1)} \equiv \left\| \begin{matrix} u_{11} & \ldots & u_{1\,m-1} \\ \ldots & \ldots & \ldots \\ u_{m\,1} & \ldots & u_{m\,m-1} \end{matrix} \right\|^{(m-1)} = 0.$$

V. *The Reduction of the Order of* $\|u_{m\,n}\|^{(m)}=0$.

§ 20. A part of the notation will be repeated at this point for convenient reference.

$$G_{kA} \equiv \|_{\overline{1,2,\ldots,k-1}}\, u_{m\,n}\|^{(m)} \equiv \left\| \begin{matrix} u_{1\,k} & u_{1\,k+1} & \ldots & u_{1\,m} & \ldots & u_{1\,n} \\ u_{2\,k} & u_{2\,k+1} & \ldots & u_{2\,m} & \ldots & u_{2\,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{m\,k} & u_{m\,k+1} & \ldots & u_{m\,m} & \ldots & u_{m\,n} \end{matrix} \right\|^{(m)},$$

$$G_{kB} \equiv \|_{\overline{1,2,\ldots,k-2}}\, u_{m\,m}\|^{(m-k+2)} \equiv \left\| \begin{matrix} u_{1\,k-1} & \ldots & u_{1\,m} \\ u_{2\,k-1} & \ldots & u_{2\,m} \\ \ldots & \ldots & \ldots \\ u_{m\,k-1} & \ldots & u_{m\,m} \end{matrix} \right\|^{(m-k+2)},$$

$$G_{kC} \equiv \|_{\overline{1,2,\ldots,k-2},\, h_1,\ldots,h_{k-2}}\, u_{m\,m}\|^{(m)} \equiv \left\| \begin{matrix} u_{1\,k-1} & \ldots & u_{1\,m} & u_{1\,h_1} & \ldots & u_{1\,h_{k-2}} \\ u_{2\,k-1} & \ldots & u_{2\,m} & u_{2\,h_1} & \ldots & u_{2\,h_{k-2}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{m\,k-1} & \ldots & u_{m\,m} & u_{m\,h_1} & \ldots & u_{m\,h_{k-2}} \end{matrix} \right\|^{(m)},$$

$$G_{kD} \equiv \|_{\overline{1,2,\ldots,k-2,}\, h_1,\ldots,h_{k-3}}\, u_{m\,m}\|^{(m-1)} \equiv \left\| \begin{matrix} u_{1\,k-1} & \ldots & u_{1\,m} & u_{1\,h_1} & \ldots & u_{1\,h_{k-3}} \\ u_{2\,k-1} & \ldots & u_{2\,m} & u_{2\,h_1} & \ldots & u_{2\,h_{k-3}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{m\,k-1} & \ldots & u_{m\,m} & u_{m\,h_1} & \ldots & u_{m\,h_{k-3}} \end{matrix} \right\|^{(m-1)},$$

$$G_k \equiv G_{kA} + G_{kB} \quad \text{and} \quad G'_k \equiv G_{kA} + G_{kC}.$$

It will be noticed that $G_1 \equiv G'_1 \equiv G_{1A}$, since $G_{1B} \equiv G_{1C} \equiv 0$; that is, no system G_{1B} or G_{1C} is formed.

No system G_{kA} can be formed when $n-(k-1) < m$; that is, when $k > n-m+1$. Likewise no system G_{kB} can be formed when $m-(k-2) < 1$; that is, when $k > m+1$. G_{kC} is a single determinant.

§ 21. Since $G_1 \equiv \|u_{m\,n}\|^{(m)}$, $m \leqq n$, contains $n-m+1$ linearly independent determinants (theorem 5), the elements of its determinants will be considered to be non-homogeneous functions of degree as previously stated (§ 2), of a set of $n-m+1$ variables; so that the system $\|u_{m\,n}\|^{(m)}$ will be exactly sufficient to determine value systems of the variables.

The determinants of G_1 can be linearly expressed in terms of

$$G_2' \equiv G_{2A} + G_{2C} \equiv \| {}_{\overline{1}} u_{m\,n} \|^{(m)} + \| u_{m\,m} \|^{(m)}$$

if $G_{3D} \equiv \| {}_{\overline{1}} u_{m\,m} \|^{(m-1)} \neq 0$. (Problem 1.) Therefore

THEOREM 6: $O_{G_1} \leqq O_{G_2}$ (note $G_2 \equiv G_2'$), where

$$G_2 \equiv G_{2A} + G_{2B} \equiv \| {}_{\overline{1}} u_{m\,n} \|^{(m)} + \| u_{m\,m} \|^{(m)}.$$

§ 22. But G_1 is not linearly expressible in terms of G_2' if $G_{3D} = 0$ (problem 1, corollory). Hence, if a system of equations can be formed from which values of the variables can be determined that will make G_{3D} and G_2 vanish simultaneously, one will have $G_2 = 0$ without having $G_1 = 0$. In that case, the order of such a system should be deducted from O_{G_2} in determining O_{G_1}. It will be shown that G_3 is such a system.

$G_{3A} \equiv \| {}_{\overline{1,2}} u_{m\,n} \|^{(m)}$ contains $n-m-1$ linearly independent determinants (theorem 5), and, therefore, to make it vanish requires the fixing of $n-m-1$ variables in terms of the remaining ones, and leaves but two to provide for the vanishing of G_{3D} and $G_2 \equiv G_{2A} + G_{2B}$. But if G_{3A} and G_{3D} vanish together, G_{3C} vanishes also, since it can be expanded in terms of G_{3D}, and therefore G_{2A} vanishes, unless $G_{4D} \equiv \| {}_{\overline{12}\,h_1} u_{m\,m} \|^{(m-1)} = 0$, since G_{2A} is linearly expressible in terms of G_3', unless $G_{4D} = 0$ (problem 1); so that the problem is to cause G_{3D} and G_{2B} to vanish by the fixing of two variables. This might seem to be impossible, since G_{3D} contains two and G_{2B} one linearly independent determinant. But the three may not be independent of each other, since the columns of G_{3D} are to be found in G_{2B}; in fact, if the two variables are so fixed as to make $G_{3D} \equiv G_{3B} = 0$, G_{2B} will vanish, since the minors of the elements of its first column vanish. Thus G_3 is a system whose vanishing uniquely determines the values of the $n-m+1$ variables, so as to make $G_2 = 0$ without making $G_1 = 0$. Therefore

THEOREM 7: $O_{G_1} \overset{=}{>} O_{G_2} - G_{G_3}$.

§ 23. But G_2 is not linearly expressible in terms of G_3' if $G_{4D} = 0$ (problem 1, corollory). Hence, if a system of equations can be formed, from which values of the variables can be determined that will make G_{4D} and G_3 vanish simultaneously, one will have $G_3 = 0$ without having $G_2 = 0$. In this case, not more than the excess of O_{G_3} over the order of such a system should be deducted from O_{G_2} in determining O_{G_1}. It will be shown that G_4 is such a system.

$G_{4A} \equiv \| {}_{\overline{123}} u_{m\,n} \|^{(m)}$ contains $n-m-2$ linearly independent determinants (theorem 5), and, therefore, to make it vanish requires the fixing of $n-m-2$ variables in terms of the remaining ones and leaves but three to provide for the vanishing of G_{4D}, G_{3A}, and G_{3B}. But if G_{4A} and G_{4D} vanish simultaneously, $G_{4C} \equiv \| {}_{\overline{12}\,h_1\,h_2} u_{m\,m} \|^{(m)}$ vanishes also, as it can be expanded in terms of G_{4D} by

Laplace's development, and therefore G_{3A} vanishes also, unless

$$G_{5D} \equiv \| \overline{123}\, h_1\, h_2\, u_{m\,m} \|^{(m-1)} = 0,$$

since G_{3A} is linearly expressible in terms of $G'_4 \equiv G_{4A} + G_{4C}$, unless $G_{5D} = 0$ (problem 1); so that the problem is, to cause G_{4D} and G_{3B} to vanish simultaneously by the fixing of three variables. This again might seem impossible, since G_{3B} contains two determinants linearly independent of each other and G_{4D} a like number. However, the four need not be linearly independent of each other, as the two matrices have some common columns. The problem can be solved by fixing the remaining three variables so as to make all the determinants of a matrix of a set of common columns (in terms of which all the determinants can be expanded by Laplace's development) vanish, provided they have a sufficient number of common columns to form a matrix containing not more than three linearly independent determinants. They are seen to have $m-2$ common columns, which is just sufficient and therefore uniquely determines the remaining three variables; that is, the $n-m+1$ variables are determined from the system $G_4 = 0$. It follows that

THEOREM 8: $O_{G_1} \leqq O_{G_2} - O_{G_3} + O_{G_4}$.

§ 24. Suppose it has been shown where k is even that

$$O_{G_1} \geqq O_{G_2} - O_{G_3} + \ldots\ldots - O_{G_{k-1}},$$

and where k is odd that

$$O_{G_1} \leqq O_{G_2} - O_{G_3} + \ldots\ldots + O_{G_{k-1}};$$

it will be shown in the former case that

$$O_{G_1} \leqq O_{G_2} - O_{G_3} + \ldots\ldots + O_{G_k},$$

and in the latter that

$$O_{G_1} \geqq O_{G_2} - O_{G_3} + \ldots\ldots - O_{G_k};$$

that is, it will be shown in both cases that the substitution of $O_{G_{k-1}} - O_{G_k}$ for $O_{G_{k-1}}$ reverses the sense of the inequality.

$G_{k-2\,A}$ is not linearly expressible in terms of G'_{k-1} if $G_{kD} = 0$ (problem 1, corollary). Hence, if a system of equations can be formed, from which values of the variables can be determined that will make G_{kD} and G_{k-1} vanish simultaneously, one will have $G_{k-1} = 0$ without having $G_{k-2} = 0$. In that case, not more than the excess of $O_{G_{k-1}}$ over the order of such a system should be deducted from $O_{G_{k-2}}$ in determining O_{G_1}. It will be shown that G_k is such a system.

G_{kA}, where $k < n-m+2$, contains $n-m-k+2$ linearly independent determinants (theorem 5), and, therefore, to make it vanish requires the fixing of $n-m-k+2$ variables in terms of the remaining ones, and leaves but $k-1$ to provide for the vanishing of G_{kD} and G_{k-1}. But if G_{kA} and G_{kD} vanish simultaneously, G_{kC} vanishes also, as it can be expanded in terms of G_{kD} by Leplace's development, and therefore G_{k-1A} also vanishes unless $G_{k+1D}=0$, since G_{k-1A} is linearly expressible in terms of G'_k unless $G_{k+1D}=0$ (problem 1); so that the problem is to cause G_{kD} and G_{k-1B} to vanish simultaneously by the fixing of $k-1$ variables. This might seem impossible, since G_{k-1B} contains $k-2$ determinants linearly independent of each other, and G_{kD} two of them. However, these k determinants need not all be linearly independent of each other, as the two matrices may have some common columns. The problem can be solved by fixing the remaining $k-1$ variables so as to make vanish all the determinants of a matrix of a set of common columns (in terms of which all the determinants can be expanded by Laplace's development), provided they have a sufficient number of common columns to form a matrix containing not more than $k-1$ linearly independent determinants. They are seen to have $m-k+2$ common columns, which, if $m-k+2>0$, is just sufficient and therefore uniquely determines the $k-1$ remaining variables; that is, the $n-m+1$ variables are determined from the system $G_k=0$. Therefore

THEOREM 9: The substitution of $O_{G_{k-1}}-O_{G_k}$ for $O_{G_{k-1}}$ reverses the sense of the inequality limiting O_{G_1} when $k<m+2$.

§ 25. But when $k=m+2$, the reduction ceases, for

$$G_{m+2D} \equiv \|_{\overline{1, 2, \ldots, m}, h_1, h_2, \ldots, h_{m-1}} u_{m\,n}\|^{(m-1)}$$

and

$$G_{m+1B} \equiv \|_{\overline{1, 2, \ldots, m-1}} u_{m\,m}\|^{(1)}$$

have no common columns, and therefore the two linearly independent determinants of the former and the $m-1$ of the latter are all linearly independent of each other, and the m remaining variables can not be so chosen as to make both sets vanish. Therefore the $n-m+1$ variables can not be so determined as to render $G_{m+1}=0$ without rendering $G_m=0$. Thus no reduction should be made from $O_{G_{m+1}}$ in calculating O_{G_1}, and therefore

THEOREM 10: $O_{G_1}=O_{G_2}-O_{G_3}+\ldots\ldots+(-1)^{m+1}O_{G_{m+1}}$.

§ 26. When $k=n-m+2$, G_{kA} can not be formed. In this case, however, the $n-m+1$ variables can be so chosen as to make G_{kB} vanish, which will in turn cause the vanishing of G_{k-1} without causing the vanishing of G_{k-2}. To express this result by the formula of theorem 10, it is to be observed that when $k=n-m+2$, $G_k \equiv G_{kB}$, and that when $k>n-m+2$, $G_k=0$. It follows for all cases that

THEOREM 11:[*] $O_{G_1}=O_{G_2}-O_{G_3}+\ldots\ldots+(-1)^{m+1}O_{G_{m+1}}$, which is a reduction formula for the order of $\|u_{m\,n}\|^{(m)}=0$.

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[*] Compare Brill, "Ueber algebraische Correspondenzen" II, *Clebsch-Neumann Mathematische Annalen*, Vol. XXXVI (1890). (Brill's paper which discusses special groups of points on a curve has been called to the writer's attention since the completion of this paper.)

§ 27. In the search for the actual order of $\|u_{m\,n}\|^{(m)}$, the order of $\|u_{m\,m+1}\|^{(m)}$ will next be investigated. It may for the present be regarded as containing two variables only, for when used in the formula of theorem 11 it will be seen to be combined with a matrix containing $n-m-1$ linearly independent determinants by the use of which the remaining $n-m-1$ variables may be expressed in terms of two of them.

§ 28. A part of the notation will here be repeated:

$$H_1 \equiv \|u_{m\,m+1}\|^{(m)} \equiv \left\| \begin{matrix} u_{11} \cdots\cdots u_{1\,m+1} \\ \cdots\cdots\cdots\cdots \\ u_{m\,1} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m)},$$

$$H_{2A} \equiv \|{}_{\overline{2}}\,u_{m\,m+1}\|^{(m)} \equiv \left\| \begin{matrix} u_{11} \; u_{13} \cdots\cdots u_{1\,m+1} \\ \cdots\cdots\cdots\cdots\cdots \\ u_{m\,1}\, u_{m\,3} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m)},$$

$$H_{2B} \equiv \|{}_{\overline{1}}\,u_{m\,m+1}\|^{(m)} \equiv \left\| \begin{matrix} u_{12} \cdots\cdots u_{1\,m+1} \\ \cdots\cdots\cdots\cdots \\ u_{m\,2} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m)},$$

$$H_{3C} \equiv \|{}_{\overline{1,2}}\,u_{m\,m+1}\|^{(m-1)} \equiv \left\| \begin{matrix} u_{13} \cdots\cdots u_{1\,m+1} \\ \cdots\cdots\cdots\cdots \\ u_{m\,3} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-1)},$$

$$H_{3A} \equiv \left\| \begin{matrix} u_{13} \cdots\cdots u_{1\,m+1} \\ u_{33} \cdots\cdots u_{3\,m+1} \\ \cdots\cdots\cdots\cdots \\ u_{m\,3} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{3B} \equiv \left\| \begin{matrix} u_{23} \cdots\cdots u_{2\,m+1} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ u_{m\,3} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{4C} \equiv \left\| \begin{matrix} u_{33} \cdots\cdots u_{3\,m+1} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ u_{m\,3} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-2)},$$

$$H_{4A} \equiv \left\| \begin{matrix} u_{33} \; u_{35} \cdots\cdots u_{3\,m+1} \\ u_{43} \; u_{45} \cdots\cdots u_{4\,m+1} \\ \cdots\cdots\cdots\cdots\cdots \\ u_{m\,3}\, u_{m\,5} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{4B} \equiv \left\| \begin{matrix} u_{34} \cdots\cdots u_{3\,m+1} \\ u_{44} \cdots\cdots u_{4\,m+1} \\ \cdots\cdots\cdots\cdots \\ u_{m\,4} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{5C} \equiv \left\| \begin{matrix} u_{35} \cdots\cdots u_{3\,m+1} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ u_{m\,5} \cdots\cdots u_{m\,m+1} \end{matrix} \right\|^{(m-3)},$$

$H_{m+1\,A} \equiv u_{m\,m}$ or $u_{m-1\,m+1}$, according as m is odd or even,

$H_{m+1\,B} \equiv u_{m\,m+1}$,

$H_{m+1\,C} \equiv \|u_{m\,m-1}\,u_{m\,m}\|^{(1)}$ or $u_{m-1\,m+1}$, according as m is odd or even,

$$H_k \equiv H_{kA} + H_{kB}.$$

§ 29. H_1 is expressible in terms of H_2 (theorem 1) unless $H_{3C}=0$. Therefore

$$O_{H_1} \leqq O_{H_2}. \quad \text{(Problem 1.)}$$

But if a system of values can be determined for the two variables such that $H_{3C}=0$, H_1 can not be expressed in terms of H_2. This is possible, as H_{3C} contains but two linearly independent determinants. So if a system of equations can be formed from which values of the variables can be determined so as to make H_{3C} and H_2 vanish simultaneously, one will have $H_2=0$ without having

$H_1=0$. H_{3C} is such a system, for H_2 can be expanded in terms of H_{3C}. Therefore

$$O_{H_1}=O_{H_2}-O_{H_{3C}}.$$

H_{3C} is a matrix like H_1 with m reduced by 1. Hence,

$$O_{H_{3C}}=O_{H_3}-O_{H_{4C}}$$

and

$$\begin{aligned} O_{H_{4C}}&=O_{H_4}-O_{H_{5C}},\\ &\dots\dots\dots\dots\dots,\\ O_{H_{mC}}&=O_{H_m}-O_{H_{m+1C}}. \end{aligned}$$

But $O_{H_{m+1C}}=O_{H_{m+1}}$. Therefore

THEOREM 12: $O_{H_1}=O_{H_2}-O_{H_3}+O_{H_4}-\dots\dots+(-1)^{m+1}O_{H_{m+1}}$.

It is to be observed that H_{kA} and H_{kB} are single determinants and that theorem 12, therefore, gives the order of $\|u_{m\;m+1}\|^{(m)}$ in terms of the orders of determinants.

VI. *The Calculation of the Order of* $\|u_{m\,n}\|^{(m)}=0$.

§ 30. By the use of the symmetric functions already defined (§ 6) this order will be calculated first when m is even and second when m is odd, $u_{r\,s}$ being considered of degree $a_s+\alpha_r$.

When m is even,

$$\begin{aligned} O_{H_1}=\,&({}_{\overline{2}}g_1+a_1+\gamma_1)({}_{\overline{2}}g_1+a_2+\gamma_1)-({}_{\overline{2}}g_1+{}_{\overline{2}}\gamma_1+\alpha_1)({}_{\overline{2}}g_1+{}_{\overline{2}}\gamma_1+\alpha_2)\\ &+({}_{\overline{4}}g_1+a_3+{}_{\overline{2}}\gamma_1)({}_{\overline{4}}g_1+a_4+{}_{\overline{2}}\gamma_1)-({}_{\overline{4}}g_1+{}_{\overline{4}}\gamma_1+\alpha_3)({}_{\overline{4}}g_1+{}_{\overline{4}}\gamma_1+\alpha_4)\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+({}_{\overline{2k}}g_1+a_{2k-1}+{}_{\overline{2k-2}}\gamma_1)({}_{\overline{2k}}g_1+a_{2k}+{}_{\overline{2k-2}}\gamma_1)-({}_{\overline{2k}}g_1+{}_{\overline{2k}}\gamma_1+\alpha_{2k-1})({}_{\overline{2k}}g_1+{}_{\overline{2k}}\gamma_1+\alpha_{2k})\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+({}_{\overline{m-2}}g_1+a_{m-3}+{}_{\overline{m-4}}\gamma_1)({}_{\overline{m-2}}g_1+a_{m-2}+{}_{\overline{m-4}}\gamma_1)-({}_{\overline{m-2}}g_1+{}_{\overline{m-2}}\gamma_1+\alpha_{m-3})({}_{\overline{m-2}}g_1+{}_{\overline{m-2}}\gamma_1+\alpha_{m-2})\\ &+({}_{\overline{m}}\,g_1+a_{m-1}+{}_{\overline{m-2}}\gamma_1)({}_{\overline{m}}\,g_1+a_m+{}_{\overline{m-2}}\gamma_1)-({}_{\overline{m}}\,g_1+{}_{\overline{m}}\,\gamma_1+\alpha_{\overline{m-1}})({}_{\overline{m}}\,g_1+{}_{\overline{m}}\,\gamma_1+\alpha_m)\\ &=A+B+C, \end{aligned}$$

where

$$\begin{aligned} A&\equiv\sum_{k=1}^{k=m/2}\left[(a_{2k-1}+a_{2k})\,{}_{\overline{2k}}g_1+a_{2k-1}\cdot a_{2k}\right]=\sum_{k=1}^{k=m/2}({}_{\overline{2k-2}}g_2-{}_{\overline{2k}}g_2)\\ &=\overline{g_2-{}_{\overline{2}}g_2}+\overline{{}_{\overline{2}}g_2-{}_{\overline{4}}g_2}+\dots\dots+\overline{{}_{\overline{2m-2}}g_2-{}_{\overline{m}}g_2}\\ &=g_2, \end{aligned}$$

and

$$\begin{aligned} B\equiv\,&\gamma_1(2\,{}_{\overline{2}}g_1+a_1+a_2)-{}_{\overline{2}}\gamma_1\cdot 2\,{}_{\overline{2}}g_1-{}_{\overline{2}}g_1(\alpha_1+\alpha_2)\\ &+{}_{\overline{2}}\gamma_1(2\,{}_{\overline{4}}g_1+a_3+a_4)-{}_{\overline{4}}\gamma_1\cdot 2\,{}_{\overline{4}}g_1-{}_{\overline{4}}g_1(\alpha_3+\alpha_4)\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+{}_{\overline{m-4}}\gamma_1(2\,{}_{\overline{m-2}}g_1+a_{m-3}+a_{m-2})-{}_{\overline{m-2}}\gamma_1\cdot 2\,{}_{\overline{m-2}}g_1-{}_{\overline{m-2}}g_1(\alpha_{m-3}+\alpha_{m-2})\\ &+{}_{\overline{m-2}}\gamma_1(2\,{}_{\overline{m}}g_1+a_{m-1}+a_m)-{}_{\overline{m}}\gamma_1\cdot 2\,{}_{\overline{m}}g_1-{}_{\overline{m}}g_1(\alpha_{m-1}+\alpha_m)\\ =\,&\gamma_1(g_1+{}_{\overline{2}}g_1-{}_{\overline{2}}g_1)-{}_{\overline{2}}\gamma_1\cdot{}_{\overline{2}}g_1\\ &+{}_{\overline{2}}\gamma_1({}_{\overline{2}}g_1+{}_{\overline{4}}g_1-{}_{\overline{4}}g_1)-{}_{\overline{4}}\gamma_1\cdot{}_{\overline{4}}g_1\\ &+\dots\dots\dots\dots\dots\dots\\ &+{}_{\overline{m-4}}\gamma_1({}_{\overline{m-4}}g_1+{}_{\overline{m-2}}g_1-{}_{\overline{m-2}}g_1)-{}_{\overline{m-2}}\gamma_1\cdot{}_{\overline{m-2}}g_1\\ &+{}_{\overline{m-2}}\gamma_1({}_{\overline{m-2}}g_1+{}_{\overline{m}}g_1-{}_{\overline{m}}g_1)-{}_{\overline{m}}\gamma_1\cdot{}_{\overline{m}}g_1\\ =\,&g_1\gamma_1, \end{aligned}$$

and
$$\begin{aligned}C\equiv\,&\gamma_1^2-{}_{\bar{2}}\gamma_1^2-[\,(\alpha_1+\alpha_2)\,{}_{\bar{2}}\gamma_1+\alpha_1\,\alpha_2]\\&+{}_{\bar{2}}\gamma_1^2-{}_{\bar{4}}\gamma_1^2-[\,(\alpha_3+\alpha_4)\,{}_{\bar{4}}\gamma_1+\alpha_3\,\alpha_4]\\&+\cdots\cdots\cdots\cdots\cdots\cdots\\&+{}_{\overline{m-4}}\gamma_1^2-{}_{\overline{m-2}}\gamma_1^2-[(\alpha_{m-3}+\alpha_{m-2})\,{}_{\overline{m-2}}\gamma_1+\alpha_{m-3}\cdot\alpha_{m-2}]\\&+{}_{\overline{m-2}}\gamma_1^2-\alpha_{m-1}\,\alpha_m\\=\,&\gamma_1^2-\sum_{k=1}^{k=m/2}[\,(\alpha_{2k-1}+\alpha_{2k})\,{}_{\overline{2k}}\gamma_1+\alpha_{2k-1}\cdot\alpha_{2k}]\\=\,&\gamma_1^2-\sum_{k=1}^{k=m/2}(\,{}_{\overline{2k-2}}\gamma_2-{}_{\overline{2k}}\gamma_2)\\=\,&\gamma_1^2-\gamma_2+{}_{\bar{2}}\gamma_2-{}_{\bar{2}}\gamma_2+{}_{\bar{4}}\gamma_2-\;\ldots\;-{}_{\overline{m-2}}\gamma_2+{}_{\bar{m}}\gamma_2\\=\,&\gamma_1^2-\gamma_2\\=\,&\delta_2\,.\end{aligned}$$

Therefore
$$O_{H_1}=g_2+g_1\gamma_1+\delta_2=K_2\,.$$

When m is odd,
$$\begin{aligned}O_{H_1}\equiv\,&(\,{}_{\bar{2}}g_1+a_1+\gamma_1)\,(\,{}_{\bar{2}}g_1+a_2+\gamma_1)-(\,{}_{\bar{2}}g_1+{}_{\bar{2}}\gamma_1+\alpha_1)\,(\,{}_{\bar{2}}g_1+{}_{\bar{2}}\gamma_1+\alpha_2)\\&+(\,{}_{\bar{4}}g_1+a_3+{}_{\bar{2}}\gamma_1)\,(\,{}_{4}g_1+a_4+{}_{\bar{2}}\gamma_1)-(\,{}_{\bar{4}}g_1+{}_{\bar{4}}\gamma_1+\alpha_3)\,(\,{}_{\bar{4}}g_1+{}_{\bar{4}}\gamma_1+\alpha_4)\\&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+(\,{}_{\overline{2k}}g_1+a_{2k-1}+{}_{\overline{2k-2}}\gamma_1)\,(\,{}_{\overline{2k}}g_1+a_{2k}+{}_{\overline{2k-2}}\gamma_1)-(\,{}_{\overline{2k}}g_1+{}_{2k}\gamma_1+\alpha_{2k-1})\,(\,{}_{\overline{2k}}g_1+{}_{2k}\gamma_1+\alpha_{2k})\\&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+(\,{}_{\overline{m-1}}g_1+a_{m-2}+{}_{\overline{m-3}}\gamma_1)\,(\,{}_{\overline{m-1}}g_1+a_{m-1}+{}_{\overline{m-3}}\gamma_1)-(\,{}_{\overline{m-1}}g_1+{}_{\overline{m-1}}\gamma_1+\alpha_{m-2})\,(\,{}_{\overline{m-1}}g_1+{}_{\overline{m-1}}\gamma_1+\alpha_{m-1})\\&+(a_m+\alpha_m)\,(a_{m+1}+\alpha_m)\\=\,&A+B+C\\=\,&g_2+g_1\gamma_1+\delta_2\\=\,&K_2\,.\end{aligned}$$

Therefore

THEOREM 13: $O_{H_1}=K_2$.

By an interchange of rows and columns it is evident that

THEOREM 14: $O_{\|u_{m+1\,m}\|^{(m)}}=J_2$.

§ 32. The order of $\|u_{m\,m+2}\|^{(m)}$ will next be investigated. It will be obtained by mathematical induction after $O_{\|u_{24}\|^{(2)}}$ and $O_{\|u_{35}\|^{(3)}}$ have been calculated.

By theorems 12, 13 and 14, and formula 5,
$$O_{\|u_{24}\|^{(2)}}=O_{\|{}_{\bar{1}}u_{24}\|^{(2)}}\cdot O_{\|u_{22}\|^{(2)}}-O_{\|{}_{\overline{12}}u_{24}\|^{(2)}}\cdot O_{\|{}_{\bar{1}}u_{22}\|^{(1)}}$$

THEOREM 15: $$={}_{\bar{1}}K_2\,J_1-{}_{\bar{2}}K_1\cdot{}_{\bar{1}}J_2=K_3\,,$$

the reduction ceasing when $k=m+2=4$; and similarly

THEOREM 16: $O_{\|u_{42}\|^{(2)}}=J_3$;
$$O_{\|u_{35}\|^{(3)}}=O_{\|{}_{\bar{1}}u_{35}\|^{(3)}}\cdot O_{\|u_{33}\|^{(3)}}-O_{\|{}_{\overline{12}}u_{35}\|^{(3)}}\cdot O_{\|{}_{\bar{1}}u_{33}\|^{(2)}}+O_{\|{}_{\overline{123}}u_{35}\|^{(3)}}\cdot O_{\|{}_{\overline{12}}u_{33}\|^{(1)}}$$

THEOREM 17: $= {}_{\overline{1}}K_2 J_1 - {}_{\overline{12}}K_1 \cdot {}_{\overline{1}}J_2 + {}_{\overline{12}}J_3 = K_3$;
and similarly

THEOREM 18: $O_{\|u_{53}\|^{(3)}} = J_3$.

It is to be noticed that in the case of $\|_{\overline{113}}u_{35}\|^{(3)}$, $k = n - m + 2 = 4$, and therefore its order is to be regarded as 1. Also it is to be noticed that the reduction ceases when $k = m + 2 = 5$ (§ 25).

Let h be a positive integer for which, and for all smaller ones, it has been verified that $O_{\|u_{h\,h-2}\|^{(h-2)}} = J_3$; then

$$O_{\|u_{h\,h+2}\|^{(h)}} = O_{\|_{\overline{1}}u_{h\,h+2}\|^{(h)}} \cdot O_{\|u_{h\,h}\|^{(h)}} - O_{\|_{\overline{12}}u_{h\,h+2}\|^{(h)}} \cdot O_{\|_{\overline{1}}u_{h\,h}\|^{(h-1)}}$$
$$+ O_{\|_{\overline{123}}u_{h\,h+2}\|^{(h)}} \cdot O_{\|_{\overline{12}}u_{h\,h}\|^{(h-2)}}$$

THEOREM 19: $= {}_{\overline{1}}K_2 J_1 - {}_{\overline{12}}K_1 \cdot {}_{\overline{1}}J_2 + {}_{\overline{12}}J_3 = K_3$.
and similarly

THEOREM 20: $O_{\|u_{h+2\,h}\|^{(h)}} = J_3$.

From theorems 16, 18, 20, it follows that

THEOREM 21: $O_{\|u_{m\,m+2}\|^{(m)}} = K_3$;
and similarly

THEOREM 22: $O_{\|u_{m+2\,m}\|^{(m)}} = J_3$.

Suppose j a positive integer for which, and for all smaller ones, it has been verified that

$$O_{\|u_{m\,m+j}\|^{(m)}} = K_{j+1} \quad \text{and} \quad O_{\|u_{m+j\,m}\|^{(m)}} = J_{j+1},$$

and note that $O_{\|_{\overline{1,2,\ldots,m-1}}u_{m\,m}\|^{(1)}} = {}_{\overline{m-1}}J_m$. Then

$$O_{\|u_{m\,m+j+1}\|^{(m)}} = O_{\|_{\overline{1}}u_{m\,m+j+1}\|^{(m)}} \cdot O_{\|u_{m\,m}\|^{(m)}} - O_{\|_{\overline{12}}u_{m\,m+j+1}\|^{(m)}} \cdot O_{\|_{\overline{1}}u_{m\,m}\|^{(m-1)}} + \ldots.$$
$$+ (-1)^{m+1} O_{\|_{\overline{1,\ldots,m}}u_{m\,m+j+1}\|^{(m)}} \cdot O_{\|_{\overline{1,\ldots,m-1}}u_{m\,m}\|^{(1)}}$$
$$= {}_{\overline{1}}K_j \cdot J_1 - {}_{\overline{2}}K_{j-1} \cdot {}_{\overline{1}}J_1 + {}_{\overline{3}}K_{j-2} \cdot {}_{\overline{2}}J_2 - \ldots.$$
$$+ (-1)^{m-1}\, {}_{\overline{m-1}}K_{j-m+3} \cdot {}_{\overline{m-2}}J_{m-1} + (-1)^m\, {}_{\overline{m}}K_{j-m+2} \cdot {}_{\overline{m-1}}J_m$$

THEOREM 23: $= K_{j+2}$ (formula 10);
and by the interchange of rows and columns,

THEOREM 24: $O_{\|u_{m+j+1\,m}\|^{(m)}} = J_{j+2}$.

It follows from theorems 13, 21 and 23, that

THEOREM 25: $O_{\|u_{m\,n}\|^{(m)}} = K_{n-m+1}$, where $n > m$; and $O_{\|u_{m\,n}\|^{(n)}} = J_{m-n+1}$, where $m > n$.

§ 33. Theorem 12 evidently provides a reduction formula for m and n, and the argument of the preceding section may seem unnecessarily extended unless it be observed that the theorem furnishes no reduction for the quantity $n - m$.

It may be seen that the greatest difference between the number of columns and the number of rows occurring in a matrix of the right member is $n - m$.

First it will be observed that this difference is less in G_{kA} than in G_{k-1A}, but greater in G_{kB} than in G_{k-1B}. This difference is therefore greatest either in G_{2A} or in G_{kB}, where k has its greatest value. In G_{2A} it is $n-m+1$. The greatest value of k is $n-m+2$ (§ 26), and

$$G_{n-m+2\,B} \equiv \|\overline{{}_{1,\,2,\,\ldots,\,n-m}}\, u_{m\,m}\|^{(2m-n)}$$

and is of the form $\|u_{m'\,n'}\|$, where $m'=m$ and $n'=m-(n-m)=2m-n$, giving $m'-n'=n-m$.

§ 34. In case all the constituents of $\|u_{m\,n}\|^{(m)}$ are of degree l, the a's may be taken each equal to l, and the α's each equal to zero, and in this case

$$K_{n-m+1} = l^{n-m+1}\, {}_nC_{n-m+1} = l^{n-m+1} \cdot \frac{n!}{(n-m+1)!\,(m-1)!}.$$

VII. *A Geometric Interpretation of* K_{n-m+1}.

§ 35. In a space of $n-m+1$ dimensions, two varieties of dimensions, k and $n-m+1-k$, intersect, in general, in a finite number of points. If the $n-m+1$ variables of G_1 be considered as the non-homogeneous coordinates of a point in a space of $n-m+1$ dimensions, G_1 will represent points. These points are among the points of intersection of the two varieties G_{2A} and G_{2B} of dimensions 1 and $n-m$ respectively. But the points G_1 do not include all the points of intersection of the two varieties. From these points of intersection must be deducted all the points of intersection of the two varieties G_{3A} and G_{3B} of dimensions 2 and $n-m-1$, respectively, except those that satisfy a certain other condition, etc. Finally it is found that G_1 represents K_{n-m+1} points.

It is to be observed that if a matrix of l-ary forms represents p points in a space of l dimensions, the same matrix of $(l+k)$-ary forms will represent a variety of order p, having k dimensions in a space of $l+k$ dimensions. Thus if G_1 had been considered as a matrix of non-homogeneous forms containing $n-m+1+k$ variables instead of $n-m+1$ of them, it would represent a variety of k dimensions in a space of $n-m+1+k$ dimensions, and in that case K_{n-m+1} would represent the order of the variety.